MODELS OF RADIATIVE TRANSFER IN FURNACES*

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Absolute rigor in the treatment of radiative transfer in furnaces is probably an unattainable goal. The engineer's problem is to choose an idealization – a mathematical model – which is consistent with the accuracy desired and/or the time available for the design. The latitude of choice is in continuous process of being widened, the objectives being the inclusion of more complete allowance for reality and – sometimes only, alas – a simplification of method. To illustrate some of the principles of radiative transfer as well as the range of models becoming available, a single furnace cross section, more or less typical, will be chosen for application of several models.

The Furnace

Furnaces are typical of enclosures in which radiation is dominant, and fuel-fired furnaces represent perhaps as complex a problem of heat transfer as exists. A furnace for transferring energy to material flowing inside tubes will be chosen, representative of that class of furnaces which includes oil-refinery tube stills, cracking coils and reformers, chemical-plant processing furnaces for catalytic operations inside tubes, and gas-fired steam boilers. Let the furnace cross section be that of Fig. 1, chosen to illustrate principles with minimum complexity of details. It is to be understood that more typical cross sections would have the tubes arranged in more complex patterns.

Tube Replacement

The first step in any tube-row furnace problem is to make use of the fact that the tube-to-tube scale is so small compared to furnace dimensions as to cause the tubes to be more or less isotropically irradiated. The tube row, and its refractory backing if any, can then be replaced by an equivalent gray plane operating at tube-surface temperature. Several steps are involved. The interchange area $\overline{S_a S_b}$ between two adjacent tubes a and b (their direct interchange per unit difference in emissive power) is given by

$$\overline{S_a S_b} = D\left\{\sin^{-1}\left(\frac{D}{C}\right) + \left[\left(\frac{C}{D}\right)^2 - 1\right]^{1/2} - \frac{C}{D}\right\}. \tag{1}$$

From this the fraction F_{PT} of isotropic radiation passing through a plane P, close to and parallel to the tubes, which is intercepted by the tubes is readily obtained. It is

$$F_{PT} = \frac{\pi D}{2C} - \frac{\overline{S_a S_b}}{C}. \tag{2}$$

If the tubes are black and their refractory backing in radiative equilibrium, the fraction of the radiation passing through P towards the tubes which is intercepted by the tubes directly as well as by reradiation or diffuse reflection from the refractory backing is $\bar{F}_{PT}$, given by

$$\bar{F}_{PT} = F_{PT}(2 - F_{PT}). \tag{3}$$

Finally, if the tubes are gray Lambert surfaces (diffusely reflecting) of emissivity ε, the fraction of the radiation passing through P which is absorbed by the tubes by all mechanisms – direct irradiation, multiple reflection between tubes and between tubes and refractory, and reradiation from refractory – may be thought of as the equivalent absorptivity or emissivity ε' of an equivalent gray plane replacing the tube-refractory backing system, given by

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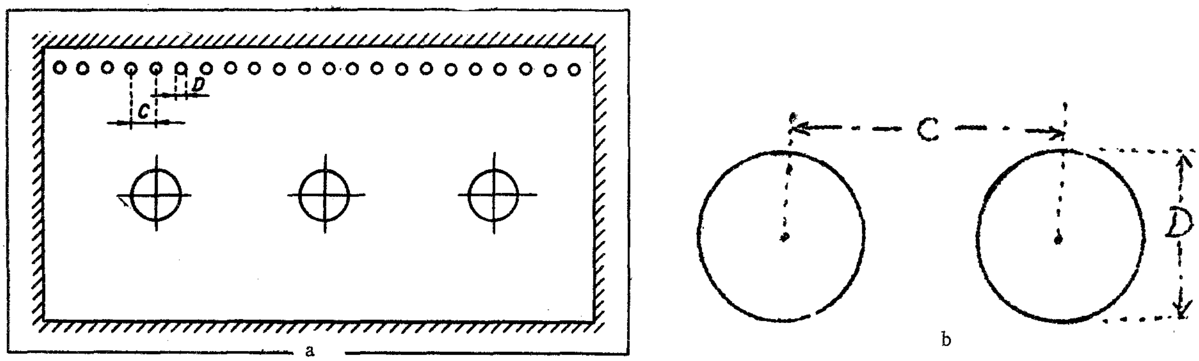

Fig. 1. a) Furnace cross section; b) roof-tube detail.

$$\varepsilon' = 1 \Big/ \left(\frac{1}{\bar{F}_{PT}} + \frac{C}{\pi D} \cdot \frac{1-\varepsilon}{\varepsilon} \right). \tag{4}$$

The derivations of Eqs. (1)-(4) and the evaluation of equivalent gray-plane emissivity for other tube arrangements and for ordered arrays of other shapes are presented elsewhere ([1], pp. 31-36; 38; 112-118; 439-441).

The Gas Model

Radiation measurements on the products of combustion, chiefly carbon dioxide and water vapor, have been made – both total-radiation measurements and spectral; and molecular models now exist which are in good agreement with both kinds of measurements. For many calculations all that is needed is a graphical representation of the total emissivity of the flue gas as a function of partial pressures of the radiating components, path length, and temperature; and a method of obtaining absorptivity from emissivity. These are available. If in the design of a furnace, however, allowance is to be made for multiple reflection and for gas temperature gradients, then total-emissivity data do not suffice. There are two alternatives. One is a complete quantitative description of the spectral details of molecular radiation from the gas component in question in the form, sometimes, of data on hundreds or even thousands of wavelength intervals, together with suitable band models. The calculation time, especially when search is for the furnace-temperature field which satisfies furnace-operating conditions, is forbiddingly large. The second and preferred approach is to represent the real-gas total-emission characteristics as an e-function series of the form

$$\varepsilon_{\text{total}} = \sum_n a_n (1 - e^{-k_n pL}), \tag{5}$$

where p is the partial pressure of radiating constituent, L is the mean beam length, k is an absorption coefficient and a is a weighting factor. This series may be visualized physically in either of two ways: as representing the a-weighted sum of n gray gases acting independently, or as representing n spectral regions or bands into which the gas is divided. Although there is an enormous variation in $k\lambda$ with wavelength λ for real gases, the restriction of interest to a moderate range of temperature and of pL permits fitting the true $\varepsilon_{\text{total}}$–T–pL relation of the gas in question with a very small number of terms – never more than three gray gases plus one clear gas (or 3 bands plus one window). Even the simplest model – one gray plus one clear gas, or one-band-plus-window model – suffices for many problems.* For that model

$$\varepsilon = a(1 - e^{-kpL}) = a(1 - e^{-kL}). \tag{6}$$

The allowance for effect of temperature can, over a moderate range, be taken care of by assuming the k's to be temperature-independent and letting $a \equiv a(T)$.

The same formulation, with the same k's, can be used to represent absorptivity α except for the fact that a is now a function not of the absorber temperature but of that of the radiation source. If α_{12} is the absorptivity of gas at T_1 for radiation from a source at T_2,

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*The one-gray-plus-clear gas model suffices, however, only for the range of pL which characterizes a particular problem; when the characteristic dimension or temperature range changes, so must k and a.

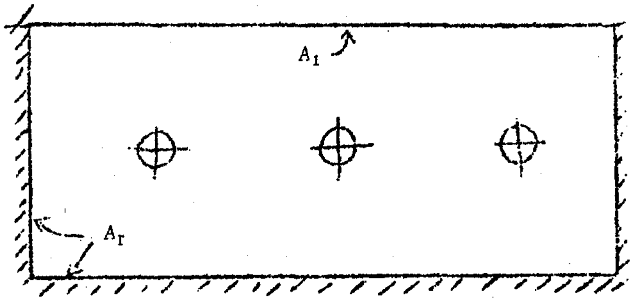

Fig. 2. Equivalent furnace cross section.

$$\alpha_{12} = \sum_n [a_n(T_2)]\,(1 - e^{-k_n L}). \tag{7}$$

Recent comparisons of radiation received at the end of a gas path along which the temperature varies widely have been made, using (a) the lengthy calculations associated with a 600-constants representation of molecular radiation, and (b) the mixed-gray gas model based on fitting the 3-gray-plus-one-clear gas model to a total emissivity–temperature–path length relation derived from the same primary source. Over all practically interesting ranges of variation in temperature along the path, and in total path length, the two methods were in excellent agreement. On the last type of problem to be considered in these notes, method (a) of supplying the gas radiation would produce an impasse.

The Well Stirred Furnace

Many industrial furnaces operate with sufficient momentum in the entering air and/or fuel to assure a reasonably well stirred combustion chamber, and to justify the assignment of single values to the composition and temperature of the radiating gases in the chamber. That assumption will be made here. The furnace, with tubes replaced, becomes a box containing heat-sink surface A_1 at known temperature T_1 controlled by the feed of the process stream, refractory surfaces designed by A_r (or by A_r, A_s, A_t if allowance is to be made for T_r variation) at an unknown equilibrium temperature T_r, and radiating gas at an unknown temperature T_g. The two unknowns T_g and heat-flux $\dot{Q}_{g\rightleftharpoons 1}$ necessitate two equations, an energy balance and a heat-transfer relation. Consider the latter.

(1) Gray Gas. A simple and useful model is based on the assumption that all the refractory is at the same unknown temperature, the sink is gray, and the gas is gray. If the sink is black, the total net interchange between gas and sink by all mechanisms – direct, multiply-reflected, and via refractory radiation – per unit difference in black emissive powers of gas and sink is called $(\overline{GS_1})_{R,\text{black}}$, having the dimensions of area and called the total-exchange area.

$$(\overline{GS_1})_{R\,\text{black}} = \varepsilon_g \left(A_1 + \frac{A_r}{1 + \dfrac{\varepsilon_g}{(1-\varepsilon_g) F_{r1}}} \right). \tag{8}$$

The view-factor F_{r1} is the fraction of all the radiation leaving A_r if black, which, without gas absorption, arrives at A_1. Since $A_1 F_{1r} = A_r F_{r1}$ and since for the present simple configuration $F_{1r} = 1$, $F_{r1} = A_1 A_r$.

When the sink is a gray Lambert surface the gas-sink flux is given by

$$Q_{g\rightleftharpoons 1} = (\overline{GS_1})_R\, \sigma (T_g^4 - T_1^4) \tag{9}$$

with

$$\frac{1}{(\overline{GS_1})_R} = \frac{1}{(\overline{GS_1})_{R\,\text{black}}} + \frac{1}{A_1}\left(\frac{1}{\varepsilon_1} - 1\right). \tag{10}$$

In this expression A_1 is the area of the plane replacing the row of tubes and their refractory backing, and ε_1 is its effective emissivity, given by (4).

The above treatment could have included allowance for variation of temperature over the refractory surfaces and for variation of the field of view of A_1, but at the expense of using matrix methods and, probably, machine computation ([1], 302-305; 368-375); and allowance could also have been made for departure

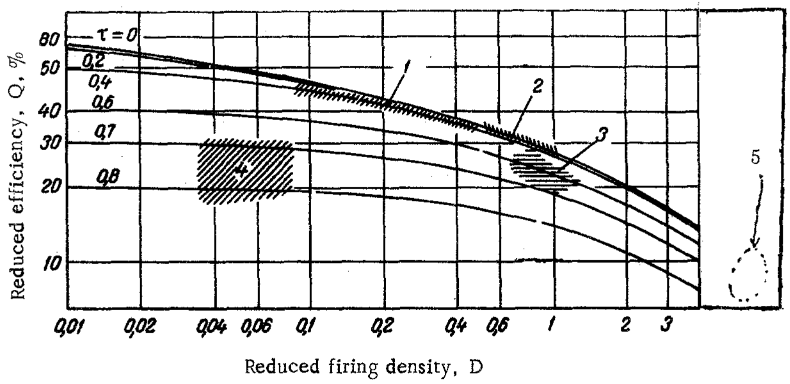

Fig. 3. Thermal performance of well stirred chambers. Reduced efficiency, Q, as a function of the reduced firing density, D', and τ: 1) radiation sections of tube stills; 2) domestic boilers; 3) soaking pits; 4) open hearths; 5) gas-turbine conductors.

from diffuse reflectance at A_1 ([1], 177-186); but the end product would in each case be a single $\overline{(GS_1)}_R$ for use in Eq. (8).

(2) Nongray Gas. In systems in which the walls are not dark gray the calculation of transmission through the zones of a system must take into account the variation of gas transmittance with wavelength. In the spectral regions of strong band absorption the radiation from a zone will be attenuated after a short passage through the gas, whereas in the spectral windows or in weak absorption bands substantially complete absorption may occur only after reflection at several walls. It is apparent that exchange between zones i and j is then inadequately described by a single S_iS_j; the quality of the radiation will change. And at refractory surfaces the energy absorbed at a wavelength of strong band emission from the gas will be reemitted as black radiation capable, in part, of passing through the windows in the gas spectrum.

Allowance for absorption-coefficient variation may be made as rigorously as desired by representation of the gas emissivity and absorptivity as those from a weighted sum of gray gases (Eq. (5)). But since a_n depends on the radiation source temperature, $\overline{GS}$ based on a's and k's will apply to a specific net direction of radiative transfer, designated by $\overrightarrow{GS}$ if from gas to surface and $\overleftarrow{GS}$ or $\overrightarrow{SG}$ if from surface to gas. Furthermore, allowance cannot be made in $\overrightarrow{GS}$ for refractory-assist, because A_r though in overall radiative equilibrium is no longer in equilibrium with respect to radiation of a specific k_i; it may be a net absorber in the spectral region of k_i and net emitter in the region of k_j. Consequently, refractory surfaces are now classed as source–sink type rather than radiative-equilibrium type. The general treatment is involved ([1], 310), but a geometrically simple idealization of great practical utility yields a simple closed-form solution. That is the case of a nongray gas (1 gray plus 1 clear), single gray sink A_1, and single gray refractory surface A_2.

The latter is here temporarily designated by A_2 – a source or sink – rather than by A_r because of its above-discussed ambivalent character. It may be shown that the net flux $\dot{Q}_{g\rightleftharpoons 1}$ is given by

$$Q_{g\rightleftharpoons 1} = \left[\overleftrightarrow{GS}_1 + \frac{1}{\dfrac{1}{\overleftrightarrow{S_1S_2}} + \dfrac{1}{\overleftrightarrow{GS}_2}}\right]\sigma(T_g^4 - T_1^4)\cdot \qquad (11)$$

The term in the bracket, allowing as it does for the radiation from gas to A_1 with the aid of the no-flux surface A_2 acting as a source in some spectral regions and sink in others, is properly designated $(\overleftrightarrow{GS}_1)_R$. The double-ended arrows indicate that, though allowance has been made for the difference in absorption and emission, the factor is applicable to flux in either direction.

A simplification comes from considering surfaces A_1 and A_r to be intimately mixed, the "speckled-furnace model." Every spot on the walls has the same view of A_1 and A_r and "sees" A_1 throughout the solid-angle fraction $A_1/(A_1 + A_r)$, designated by C_s, the "cold-surface" fraction of the furnace walls. Though the speckled-furnace assumption is a rather poor one for the present furnace, it is quite good for many furnaces in which the heat sinks, adding to A_1, are distributed over several wall, roof, and floor areas. For a speckled enclosure confining a nongray gas represented by a one-gray-plus-one-clear gas or one-band-plus-window model, and with subscript r replacing 2, $(\overleftrightarrow{GS}_1)_R$ is given by

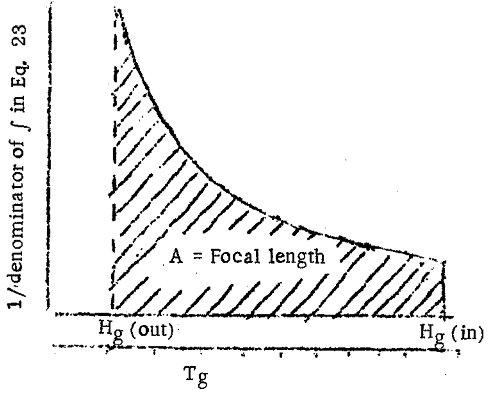

Fig. 4.

$$(\overleftrightarrow{GS}_1)_R = \frac{A_T}{\dfrac{1}{C_s\varepsilon_1 + (1-C_s)\,\varepsilon_r}\left(\dfrac{1}{a_g}-1\right) + \dfrac{1}{\varepsilon_{g,e}} + \dfrac{1}{C_s\varepsilon_1} - \dfrac{1}{a_g}} . \tag{12}$$

In this, A_T is the total envelope area of the furnace chamber $A_1 + A_r$, a_g is the value of a corresponding to the gas temperature, and $\varepsilon_{g,e}$ is the equivalent gray emissivity. The last term comes from dodging the need to express $\overrightarrow{GS}$ and $\overleftarrow{GS}$ separately; this can be done provided ε_g is evaluated at the arithmetic mean of Tg and T_1 and then multiplied by a factor $(4 + a' + b - c)/4$ where a' (unrelated to a or a_g) $= \partial \ln \varepsilon_g / \partial \ln pL$, $b = \partial \ln \varepsilon_g / \partial \ln T_g$, $c = 0.65$ for CO_2, 0.45 for H_2O, 0.5 for average flue gas. The factor may be ignored and ε_g, evaluated at T_g, used instead when $T_1 < T_g/2$. The value of a_g in any case is obtained from fitting the emissivity ε_g, at the mean-beam length L_m of the system and at twice or three times that length, to the gray-plus-clear gas model to give

$$a_g = \frac{\varepsilon^2_{g,L_m}}{2\varepsilon_{g,L_m} - \varepsilon_{g,2L_m}} . \tag{13a}$$

$$a_g = \frac{2\varepsilon_{g,\ L_m}}{3-[4(\varepsilon_{g,3L_m}/\varepsilon_{g,L_m})-3]^{1/2}} \quad \text{for } L_m \text{ and } 3L_m \tag{13b}$$

The derivations of Eqs. (11) and (13a) appear elsewhere ([1], 317-320; 250).

Equation (12) is relatively simple considering the complexity of the radiative processes for which it makes allowance.

<u>(3) Gray Gas, Speckled Walls.</u> If the simpler gray gas model represented by Eqs. (8) and (10) is converted to the speckled-wall model ($F_{r1} \rightarrow C$), those equations yield

$$\overline{(GS_1)}_R = \frac{A_T}{\dfrac{1}{C_s\varepsilon_1} + \dfrac{1}{\varepsilon_g} - 1} ; \tag{14}$$

a relation of extraordinary simplicity. Equation (14) will be used for combination with an energy balance. Let H_F represent the enthalpy in the feed stream (air and fuel) entering the chamber per hour, measured above a base T_0. For this simplified treatment the external losses from the furnace and the internal convection to the sink and refractory surfaces will be assumed negligible. An energy balance then yields

$$\frac{H_F - Q_{g\rightleftarrows 1}}{H_F} = \frac{T_g - T_0}{T_{AF} - T_0} . \tag{15}$$

This is both an energy balance and a definition of T_{AF}, which is that adiabatic flame temperature $(H_F/\dot{m}C_p)$ obtained by ignoring dissociation and by using, over the interval T_0 to T_{AF}, the same mean heat capacity of the gases as is applicable over the interval T_0 to the outlet gas temperature T_g. Elimination of T_g between (14) and (15) and dropping of subscripts on $\dot{Q}$ yields

$$\frac{Q}{\sigma(\overline{GS_1})_R} + T_1^4 = T_{AE}^4 \left(1 - \frac{Q(T_{AF} - T_0)}{H_F T_{AF}}\right)^4. \tag{16}$$

Let the following dimensionless groups be defined: $\dot{Q}'$ the reduced furnace efficiency, the actual efficiency $\dot{Q}/H_F$ times the temperature ratio $(T_{AF} - T_0)/T_{AF}$; D' the reduced firing density; $H_F/(\sigma(\overline{GS}_1)_R T_{AF}^3 (T_{AF} - T_0))$ the ratio of energy input to a kind of radiating ability; τ the ratio of sink temperature T_1 to pseudo-adiabatic flame temperature T_{AF}.

In terms of the new groups, (16) becomes

$$Q'D' + \tau^4 = (1 - Q')^4. \tag{17}$$

Furnace efficiency is seen to be a function only of firing density and relative heat-sink temperature, and the firing density term makes due allowance for such operating variables as fuel type or excess air or air preheat which affect flame temperature and gas emissivity, for fractional occupancy of the walls by sink surfaces, and for wall emissivity. The function represented by (17) is plotted in Fig. 3, which permits several interesting conclusions:

1. As the firing rate D' decreases, efficiency rises and approaches $1 - \tau$ in the limit.
2. Changes in sink temperature are unimportant if $\tau < 0.3$.
3. As the furnace-chamber wall approaches complete coverage by a black sink, i.e., as $C_S \varepsilon_1$ approaches 1, the effect of flame emissivity on D' becomes one of inverse proportionality. At extremely high firing rates where $\dot{Q}'$ becomes inversely proportional to D', the efficiency varies directly as gas emissivity ε_g; but at low firing rates, characteristic of many furnaces, the effect of ε_g is much less.
4. When the furnace wall is well covered with refractory surfaces and/or the sink is not black ($C_S \varepsilon_1 \ll 1$), a change of flame emissivity produces a much less than proportional change in heat transfer.

Equation (17) and/or Fig. 3 constitute a basis for classification of furnace types, some of which are indicated by the range of variables in which they operate.

When the furnace is not so well stirred as to justify the assumption that the enthalpy temperature of the leaving gases is the same as the mean radiating gas temperature, allowance can be made by assigning a value Δ to the difference

$$\Delta = T'_{g,\text{radiating}} - T_{g,\text{ leaving system}}. \tag{18}$$

This has the effect of inserting $+\Delta/T_{AF}$ inside the bracket of Eq. (16) or $\Delta'(\equiv\Delta/T_{AF})$ inside the parenthesis of (17). It has been found empirically that a value of 250-300°F suffices to make the modified equations fit remarkably well the data on a number of furnaces characterized by a rather high $C'_S \varepsilon_1$ (above 0.5). Rigorous calculations of far more sophisticated furnace models produce results which are illuminatingly analyzed by plotting on the coordinates of the figure representing (17) ([1], 463-470).

The Long-Furnace Model without Axial Radiative Flux

We have so far restricted interest to furnaces so well stirred that a single temperature can characterize the radiating gases. The other extreme is the long furnace, for which a cross section normal to the gas-flow direction will again be represented by Fig. 1. As before, the first step is replacement of tubes and their backwall by an equivalent gray plane, to give the cross section in Fig. 2. This furnace system is characterized by three assumptions:

1. combustion occurs so rapidly compared to total gas residence time in the furnace that the gas temperature at the burner end is the adiabatic flame temperature (though this assumption can be modified by using data on actual furnaces);
2. the furnace length in the direction of gas flow is so great compared to its height or width that net radiative flux in the x-direction (gas-flow direction) may be ignored relative to flux normal to it;
3. the gas at any flow cross section has a mean radiating temperature and mean enthalpy temperature which are the same.

From these assumptions it follows that the local flux density g at the sink at the downstream distance x is expressible in terms of local gas and sink temperatures and the local configuration – the shape in cross section and the disposition of sink and refractory surfaces. Thus $q_{g\rightleftharpoons 1} \equiv q_{g1}(T_g, T_1, P_1)$, where P_1, the perimeter occupied by the sink furnaces, is a partial measure of the shape at x. Similarly, the local external losses to surroundings at T_0 may be represented by $q_{g0} \equiv q_{g0}(T_g, P_r)$, where P_r is the perimeter occupied by refractory surfaces from which there is an external loss. In the length dx the net energy flux from the gas $d\dot{Q}_{g,\text{net}}$ is given by

$$dQ_{g,\text{net}} = q_{g\rightleftharpoons 1}P_1dx + q_{g\rightleftharpoons 0}P_rdx = -dH_g,$$

where $\dot{H}_g$ is the enthalpy per hour in the furnace gases passing a point $(\equiv \dot{m}C_pdT_g)$, and

$$q_{g\rightleftharpoons 1}P_1 = \frac{(\overline{GS_1})_R}{L}\sigma(T_g^4 - T_1^4) + U_1P_1(T_g - T_1). \tag{19}$$

The last term represents convection to the sink, and $(\overline{GS}_1)_R/L$ represents the total-exchange area per unit length of furnace. This exchange area is calculated for a three-dimensional system, as though a spot on the sink sees upstream and downstream as well as across; but its downstream view is as much colder than its cross view as its upstream view is hotter, and it receives radiation as though the whole furnace temperature pattern is the same as the temperature pattern at the cross section at x. The term $(\overline{GS}_1)_R$ can be calculated with any degree of rigor desired; in the present example it could be assumed given by Eq. (12), with $A_T/L = P_1 + P_r$, and $C_s = P_1/(P_1 + P_r)$.

If the sink temperature changes significantly along the furnace – and in a "long" furnace it generally does – another equation is necessary. If in addition the mean enthalpy temperature of the process stream inside the tubes is different than T_1, the tube skin temperature, still another relation is needed (this is the case, for example, in catalyst-packed tubes). Let the hourly enthalpy of the sink $\dot{H}_1$ be a known function of its mean enthalpy temperature T_1'; and let the overall heat-transfer coefficient U_1' from tube outer skin T_1 to process stream temperature T_1' be a known function of T_1'. The system of equations needed for a solution is then

$$\frac{-dH_g(T_g)}{dx} = \frac{(\overline{GS_1})_R}{L}\sigma(T_g^4 - T_1^4) + U_1P_1(T_g - T_1) + U_0P_r(T_g - T_0), \tag{20}$$

$$\frac{+dH_1(T_1')}{dx} = \frac{(\overline{GS_1})_R}{L}\sigma(T_g^4 - T_1^4) + U_1P_1(T_g - T_1), \tag{21}$$

$$U_1'P_1(T_1 - T_1') = \frac{(\overline{GS_1})_R}{L}\sigma(T_g^4 - T_1^4) + U_1P_1(T_g - T_1), \tag{22}$$

U_0 is the overall heat-transfer coefficient through the refractory, gas to ambient air around the furnace.* The sign of dH_1/dx indicates application to co-current flow of combustion gases and process stream.

Rigorous solution of two nonlinear simultaneous equations (20) and (21) in three unknowns, with nonlinear equation (22) used to eliminate T_1, would normally require machine computation. Inspection of Eq. (20) indicates, however, that it is relatively insensitive to changes in T_1 because of the T_g^4 term. This suggests that the relation between T_g and T_1 be guessed, and the guess fed into (20), which can then be written in the form

$$\int_0^x dx = \int_{H_{g,in}}^{H_{r}} \frac{-dH_g(T_g)}{\frac{(\overline{GS_1})_R}{L}\sigma(T_g^4 - T_1^4) + U_0P_r(T_g - T_0) + U_1P_1(T_g - T_1)}. \tag{23}$$

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*The validity of expressing the heat loss from gas to outside air in terms of an overall coefficient, with refractory temperature not mentioned, is conditional on the radiative equilibrium and gas-convective-equilibrium inside-refractory temperatures being the same. This is only true if convection, gas to refractory surface, equals conductive loss through it. Such an assumption is generally justified. If it is not made, $(\overline{GS}_1)_R$ cannot be used because A_r changes from a radiative equilibrium to a source–sink type surface; one more unknown is added, the temperature T_r; and an additional equation is needed. This will be an energy balance on A_r. Two new total-exchange areas must be evaluated, $\overline{GS}_r$ and $\overline{S_1S}_r$, and the enormous increase in complexity is seldom justified.

For every T_g the right side is calculable, and graphical integration from H_g (entering) to H_g gives the corresponding furnace position x. Three points on the curve (Fig. 4) often suffice to determine the total length of a furnace with flue gases leaving at a specified $T_{g,out}$. With the T_g–x relation established, Eq. 21 can be integrated to obtain T_1' vs T_g, with which knowledge (22) gives a new relation between T_g and T_1. When T_1 has a larger influence on (20), the above sequence is applied to small step changes in T_g, with convergence established before the next step is taken.

Allowance for Effect of Gas Temperature Gradients on Radiation

In the models discussed so far, gas–surface exchange has been assumed to occur between a surface and a uniform-temperature gas, with no intervening gas present at another temperature. Allowance for gas temperature gradients enormously complicates the problem, and the engineer should think twice before deciding on the need for more sophisticated models than those presented above. But if progress is to be made in determining, without building and rebuilding furnaces, what the effects of their operating and design variables will be on the distribution of heat fluxes to the sink surfaces, there is no alternative to a more nearly rigorous approach.

Consider an enclosure with gas and surface temperature varying throughout. A volume element dV_i will emit radiation of wavelength λ in all directions at the rate $4k_\lambda E_{\lambda,j} dV_j$ per wavelength interval, where k_λ is the monochromatic absorption coefficient, and E_λ is the monochromatic emissive power of a black body – the Planck function. The resulting flux density r-distant would be, without attenuation, $1/\pi r^2$ if taken normal to r; and the transmittance will be $e^{-\int k_\lambda(r)dr}$, the term $k_\lambda(r)$ indicating that k_λ is expected to vary along the path. The monochromatic intensity leaving a surface element dA_j will be $W_{\lambda,j}/\pi$, where W_λ is the hemispherical leaving-flux density due to both emission and reflection. At an angle θ_{ji} with the normal to dA_j and a distance r from it, the flux density due to dA_j will be reduced by the factor $(\cos\theta_{ji}/r^2)$ times the transmittance previously evaluated. From these principles one can write an energy balance on a volume dV_j within the total volume V enclosed by area A, which will include as inputs the emission from all volume elements dV_j and all surface elements dA_i and absorption by dV_i, convection from any surface touching

SCHEME 1. Energy Balance on Volume dV_i

emission to dV_i from gas elements dV_j

$$dV_i \int_0^\alpha \int_V \frac{4k_{\lambda,j} E_{\lambda,j}\, e^{-\int_0^{r_{ij}} k_\lambda(r)dr}}{4\pi r_{ij}^2}\, dV_j k_{\lambda,i}\, d$$

emission to dV_i from surface elements dA_j

$$+\, dV_i \int_0^\alpha \int_W \frac{W_{\lambda,j}\, e^{-\int_0^{r_{ij}} k_\lambda(r)dr}}{\pi r_{ij}^2} \cos\theta_{ji} dA_j k_{\lambda,i}\, d\lambda$$

$$+hdA_k(T_k - T_i) - dV_i \nabla\rho(i_s + i_c)\vec{U}$$

convection from any touching surface dA_k

net enthalpy flux, sensible and chemical into dV_i

$$= \left[dV_i \int_0^\alpha 4k_{\lambda,i} E_{\lambda,i}\, d\lambda + dV_i \frac{\partial}{\partial t}[\rho(i_s + i_c)] \right]$$

radiation emitted by dV_i

rate of increase of enthalpy in dV_i

SCHEME 2. Energy Balance on Surface dA_i

radiation emitted from dV_j's and absorbed at A_i

$$dA_i \int_0^\infty \int \frac{k_{\lambda,j} E_{\lambda,j}\, e^{-\int_0^{r_{ij}} k_\lambda(r)dr} \cos\theta_{ij} dV_j \varepsilon_{\lambda,i}\, d\lambda}{\pi r_{ij}^2}$$

absorption of radiation leaving surface dA_j

$$+dA_i \int^\infty \int_A \frac{W_{\lambda,j}\, e^{-\int^r k_\lambda(r)dr} \cos\theta_{ij}\cos\theta_{ji}\, dA_j \varepsilon_{\lambda,i}\, d\lambda}{\pi r_{ij}^2}$$

$$+h(T_j - T_i)\, dA_i$$

convection to dA_i

$$= \left[dA_i \int_0^\alpha \varepsilon_{\lambda,i} E_{\lambda,i}\, d\lambda + q_{\text{net},i}\, dA_i + dA_i \frac{\partial H_i}{\partial t} \right]$$

emission by surface

net flux extracted through surface

enthalpy change with time

dV_i, enthalpy flux into dV_i; and as outputs the radiative emission from dV_i and, for the unsteady state, its rate of energy absorption by enthalpy increase. Scheme 1 shows such a balance.

Similarly, an energy balance may be written on a surface element dA_i, Scheme 2.

The two equations are integro-differential equations, i.e., functions inside the integrals are temperature-dependent, and the temperature field is initially unknown. In the pre-digital computer days the simultaneous solution of these equations for a complex configuration would not have been considered feasible. Today there are two broad approaches to the problem: one is to think of the volume and surface as each being divided into many small isothermal zones, some of the zones having an unknown temperature and known flux, others a known temperature and unknown flux. An energy balance may be written for every zone of unknown temperature, yielding a system of nonlinear equations, in the limit an infinite matrix the solution of which is a solution of the integral equations. Solution of a finite-zone system yields a close approximation to the desired temperatures; these permit evaluation of the unknown fluxes – the usual ultimate objective of the analysis. There are no limits on the complexity of furnace geometry or of gas flow pattern in the application of the method, and it is certain to find increasing use in making furnace design more quantitative ([1], 365-377; 470-487).

The other way to approach the problem is to think not of zones and temperatures but of the radiation field, the monochromatic intensity variation along every direction at every point for every wavelength. Formulation of the variation of intensity along one of these monochromatic radiation streams would then yield a differential equation; and the whole field would be described by a multiply infinite number of such differential equations. Only for cases in which some sort of symmetry exists – for example, flux through layered slabs or along systems with dominant temperature gradients in a single direction – is the concept useful. It was suggested by Schuster and Schwarzschild many years ago for problems in astrophysics; it underlies the two-flux method of obtaining gross approximations to scatter problems in slabs. An application appeared by Roesler [2] on radiative transfer in furnaces, and the next subdivision here is based largely on his paper.

The Long-Furnace Model with Axial Radiative Flux

Consider again a long furnace of cross section as in Fig. 1 and equivalent cross section as in Fig. 2, with P_1 and P_r representing heat-sink and refractory-surface perimeters; and let the gas temperature and sink temperature vary only with downstream distance x, as in the previous long-furnace treatment. For maximum simplicity consistent with allowance for the radiation not being gray, let the gas radiation be represented by the one band-plus-window model, Eq. (6)

$$\varepsilon_g = a\,(1 - e^{-kL}).$$

This divides black-body radiation into the fraction a lying in the region of gas absorption and emission, and the fraction $1 - a$ lying in the clear-gas portion or spectral window. Let the heat sink and refractory surfaces be gray, as before, with emissivities ε_1 and ε_r.

Let the downflowing hot gas at T_g produce an enthalpy flux of $\dot{H}_g(T_g)$ or $\dot{m}_g C_p T_g$. Let the furnace be co-current, with downward flux of process-stream enthalpy in the tubes designated by $\dot{H}_1(T_1')$. The mean process-stream temperature T_1' is lower than the outer tube-skin temperature T_1, and they are connected by the relation

$$q_{1,\,net} = U_1'\,(T_1 - T_1') \tag{24}$$

where U_1' is the overall coefficient of heat transfer from outer tube-skin to main body of process stream. U_1' is assumed dependent only on the stream temperature T_1' (it must be remembered that U_1' must be adjusted for being based on an artificial sink area, $C/\pi D$ times the true tube area).

Let the furnace gases lose heat by convection to the refractory surfaces at the same rate as the conduction losses through the refractory walls and on to the ambient atmosphere at T_0, thereby making the loss rate per unit furnace height $P_r U_0 (T_g - T_0)$, where U_0 is the overall heat-transfer coefficient, gas to outside air. Let the furnace gases lose heat by convection to the tubes at the rate $U_1 P_1 (T_g - T_1)$, U_1 being adjusted the same way as U_1'.

Let the gas-flow cross section of the furnace be S. Visualize a stream of radiation in the axial direction, divided into an upward stream and a downward stream and further divided spectrally into radiation lying in the band-fraction a and the window-fraction $1 - a$ of the black-body spectrum. There are then six streams carrying energy along the x-direction, the hot gases carrying it down at the rate $\dot{H}_g$, the process stream carrying it down at the rate $\dot{H}_1$, the downwardly directed band-radiation stream at a rate designated B_D, the upwardly directed band-radiation stream designated by B_U, and the two window-radiation streams designated by W_D and W_U. These streams interact more or less with each other, with the tube surfaces P_1 and with the refractory surfaces P_r; and the problem is to formulate the space rate of change of each stream due to coupling with the others. The space rate of change $d\dot{H}_g/dx$ of the gas stream, for example, will be due to its emission into the band-radiation streams B_D and B_U and its absorption from them. The space rate of change of the downward band-radiation stream dB_D/dx will be due to its absorption of radiation from the gas, its emission, its indirect reception from the upstream band stream via reradiation from the refractory, its absorption of radiation emitted by the tubes, and its reception from the window-radiation streams W_D and W_U via their absorption by the refractory surface and reemission partially in the band region of the spectrum. The six resulting differential equations have been arranged in tabular form with $d\dot{H}_g/dx$, dB_D/dx, etc. along the left, the quantities or streams from which they receive energy along the top and the coefficients of those quantities, or coupling terms, in the main body. The construction of Scheme 3 is based on the following principles:

a) the emission rate from a gas = $4kVE_B = 4kV\sigma T_g^4$ where V is the volume;

b) since an energy balance on a section Sdx through the furnace in steady-state yields the relation

$$d(\dot{H}_g + B_D + B_U + W_D + W_U + \dot{H}_1 + \dot{Q}_{loss})/dx = 0,$$

the sum of the coupling terms in each column must be zero;

c) at equilibrium

$$B_D = B_U = a\sigma T_g^4 S,$$

$$W_D = W_U = (1 - a)\sigma T_g^4 S;$$

and at equilibrium each derivative term must equal zero, i.e., the sum of the products of the column headings, at equilibrium, by the coupling terms in a single row must become zero;

d) the coupling term for interaction of $d\dot{H}_1/dx$ or $d\dot{H}_g/dx$ with B or W is the same for both directions, D and U.

With these principles, there follows a sequence of steps for filling in the coupling terms:

1. Row 1, column 1. Since the volume corresponding to unit height is S, and the gas emits only in the energy fraction a of the spectrum, the gas stream emits $4kaS\sigma T_g^4$ and the coupling term is $4kaS$.
2. In Row 1, the above emission is exclusively, and equally, to streams B_U and B_D. Then the coefficients on B_U and B_D that satisfy (c) above are 2kS, each.
3. Since in the σT_g^4 column the only terms coupled to σT_g^4 besides $d\dot{H}_g/dx$ are the dB/dx's acting equally, the coefficient on each dB/dx coupling it to σT_g^4 is $2kaS$, to satisfy (b).
4. The loss rate $d\dot{H}/dx$ from the process stream due to direct radiation from the tube surface is $\varepsilon_1 P_1 \sigma T_1^4$; so the coefficient in the σT_1^4 column is $-\varepsilon_1 P_1$.
5. The only other streams affecting $d\dot{H}_1/dx$ are the four streams B and W, and they act equally. Then, to satisfy (c), each of the four coefficients is $\varepsilon_1 P_1/2$.
6. Consider the σT_1^4 column. Coupling terms additional to that in the $d\dot{H}_1/dx$ row will appear in the dB/dx and dW/dx rows only. Those affecting dB/dx will be proportional to a, the band fraction, those affecting dW/dx to $(1 - a)$, the window fraction; and there is no difference between down and up. Then, to satisfy (b), the coefficients must be $a\varepsilon_1 P_1/2$ for each dB/dx and $(1 - a)\varepsilon_1 P_1/2$ for each dW/dx.
7. In the dB_D/dx row, $2kaS$ in the σT_g^4 column must be balanced by a term which can only be in the B_D column, and to satisfy (c) the term must be $-2kS$. Also to satisfy (c) the term $a\varepsilon_1 P_1/2$ in the σT_1^4 column must be balanced by $-\varepsilon_1 P_1/2$ in the B_D column. But that part of the rate of change of downward band flux which is due to B_D itself is equally related to tubes and refractory; so if $-\varepsilon_1 P_1/2$ is present, so must $-\varepsilon_r P_r/2$ be. In addition, in the same box there must be $-2kS$ to offset the $+2kaS$ in the same row under σT_g^4 and thereby satisfy (c).

8. By symmetry, if $-2kS$, $-\varepsilon_1 P_1/2$, $-\varepsilon_r P_r/2$ is present at the intersection of dB_D/dx and B_D, the same must be present at the intersection of $-dB_U/dx$ and B_U.

9. In the dW_D/dx row, since $(1-a)\varepsilon_1 P_1/2$ appears as the coefficient of $\sigma T_1'^4$, a term to offset it must appear in the same row, logically in the W_D column, and to satisfy (c) the term must be $-\varepsilon_1 P_1/2$. But if the window-radiation stream is decreased by its own direct flux to P_1 there must be an equivalent direct flux to P_r; so $-\varepsilon_r P_r/2$ belongs in the W_D column as well.

10. By the same argument or by symmetry, the same terms $-\varepsilon_1 P_1/2$, $-\varepsilon_r P_r/2$ must appear, if in the $dW_D/dx - W_D$ intersection, then also in the $-dW_U/dx - W_U$ intersection.

11. In the B_D column and the dB_D/dx row the coefficient $-\varepsilon_r P_r/2$ represents loss from the B_D stream to the refractory surface. The latter gives this up to all four radiation streams, equally to B_U and back to B_D in proportion to a, and equally to $W_D + W_U$ in proportion to $(1-a)$ (corresponding to refractory-absorbed band radiation reemitted as window radiation). To make the four coefficients add to $\varepsilon_r P_r/2$ and satisfy (b), they are $a\varepsilon_r P_r/r$ for each dB/dx and $(1-a)\varepsilon_r P_r/4$ for each dW/dx in the B_D column.

12. The same argument puts the same four coefficients in the next (B) column.

13. Symmetry puts the same four coefficients in each of the next two $(W_D + W_U)$ columns; and this also satisfies (c).

This completes the set of equations, except for the obvious terms in the bottom row and last three columns, representing convection. Roesler says that in a typical steam reformer, convection to the tubes amounts to no more than one per cent of the total transfer.

The formulation of equations is necessarily inexact in several respects. For example, (1) flux from and to the gas and to and from the B_U and B_D streams is not the full 2kS due to boundary wall effects. The error is greater the smaller $k\sqrt{s}$. The effect is to reduce axial flux. (2) Radiation leaving W_D for the tubes and not absorbed is assumed to continue downwards in W_D, whereas by diffuse reflection some of it joins W_U.

To find a solution to the set of equations it is necessary to fix as many boundary conditions as equations. Comments on these and on additional requirements for solution follow:

1. $\dot{H}_g$ and H_1 are known at $x = 0$ because feed conditions are specified. $T_g(0)$ would however be the adiabatic flame temperature T_{AF} if it were assumed that combustion were instantaneous. Roesler assumes instead that there is a penetration depth L in which burner jet mixing assures a much lower initial temperature; and the discrepancy between $\dot{H}_g$ and T_g in that range is handled by putting in an artificial source function, added to the expression for $d\dot{H}_g/dx$, and corresponding in effect to delayed conversion of chemical to sensible energy and/or back-mixing. The exact form of the source function is not critical, only the magnitude of the difference between the $\dot{H}_g$ of the feed stream and the $\dot{H}_g$ corresponding to the chosen initial gas temperature $T_g(0)$.

2. The difference between T_1 and T_1' necessitates introducing an additional condition. The flux density of radiation incident on a wall at any level x is the arithmetic mean of $(B_D + W_D)/S$ and $(B_U + W_U)/S$, from which the net flux density at the tube skin can be expressed two ways, giving

$$(T_1 - T_1')\,U_1 = \varepsilon_1\left(\frac{B_D + B_U + W_D + W_U - \sigma T_1'^4}{2S}\right) + U_1(T_g - T_1) \tag{25}$$

3. The radiant streams are inter-related at the refractory ends of the system. When B_U strikes the top, $(1-\varepsilon_r)$ is reflected to put $(1-\varepsilon_r)B_U$ into stream B_D; and ε_r is absorbed and reradiated, putting $a\varepsilon_r B_U$ into B_D and $(1-a)\varepsilon_r B_U$ into W_D. When W_U strikes the top $(1-\varepsilon_r)W_U$ is reflected into W_D, $\varepsilon_r W_U$ is absorbed and reradiated, $a\varepsilon_r W_U$ into B_D and $(1-a)\varepsilon_r W_U$ into W_D. From this, B_D and W_D are known if B_U and W_U are given.

$$B_D = (1-\varepsilon_r)\,B_U + a\varepsilon_r\,(B_U + W_U) \tag{26}$$

$$W_D = (1-\varepsilon_r)\,W_U + (1-a)\,\varepsilon_r\,(W_U + B_U) \tag{27}$$

Similarly at the furnace bottom, B_U and W_U can be found if B_D and W_D are given; interchange subscripts U and D in (26) and (27).

SCHEME 3. Coupling Coefficients in the Set of Simultaneous Differential Equations

	σT_g^4	B_D	B_U	W_D	W_U	σT_1^4	T_g	T_o	T_1
dH_g/dx	$-4\kappa aS$	$2\kappa S$	$2\kappa S$	0	0	0	$-P_rU_o$ $-P_1U_1$	P_rU_o	P_1U_1
dB_D/dx	$2\kappa aS$	$-2\kappa S - \frac{\varepsilon_1 P_1}{2} - \frac{\varepsilon_r P_r}{2} + \frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_1 P_1}{2}$	0	0	0
$-dB_U/dx$	$2\kappa aS$	$\frac{a\varepsilon_r P_r}{4}$	$-2\kappa S - \frac{\varepsilon_1 P_1}{2} - \frac{\varepsilon_r P_r}{2} + \frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_r P_r}{4}$	$\frac{a\varepsilon_1 P_1}{2}$	0	0	0
dW_D/dx	0	$\frac{(1-a)\varepsilon_r P_r}{4}$	$\frac{(1-a)\varepsilon_r P_r}{4}$	$-\frac{\varepsilon_1 P_1}{2} - \frac{\varepsilon_r P_r}{2} + \frac{(1-a)\varepsilon_r P_r}{4}$	$\frac{(1-a)\varepsilon_r P_r}{4}$	$\frac{(1-a)\varepsilon_1 P_1}{2}$	0	0	0
$-dW_U/dx$	0	$\frac{(1-a)\varepsilon_r P_r}{4}$	$\frac{(1-a)\varepsilon_r P_r}{4}$	$\frac{(1-a)\varepsilon_r P_r}{4}$	$-\frac{\varepsilon_1 P_1}{2} - \frac{\varepsilon_r P_r}{2} + \frac{(1-a)\varepsilon_r P_r}{4}$	$\frac{(1-a)\varepsilon_1 P_1}{2}$	0	0	0
$d\dot{H}_1/dx$	0	$\varepsilon_1 P_1/2$	$\varepsilon_1 P_1/2$	$\varepsilon_1 P_1/2$	$\varepsilon_1 P_1/2$	$-\varepsilon_1 P_1$	P_1U_1	0	$-P_1U_1$
$\frac{d\dot{Q}_{Loss}}{dx}$	0	0	0	0	0	0	P_rU_o	$(-)$ P_rU_o	0

4. A first choice of $T_g(0)$ is made satisfactorily by assuming that its space rate of change at $x = 0$ is negligible, that the gas is consequently in equilibrium with the local band radiation stream, or that

$$\sigma T_g^4(0) = \frac{B_D(0) + B_U(0)}{2aS} \quad .$$

5. The equations are nonlinear and numerical integration is the only possibility. Roesler's comments on the difficulties of solution are illuminating. Values for B and W at the furnace top were chosen for a first trial, satisfying Eqs. (26) and (27). The intention was to march down the furnace with all six equations (he did not include any convection in his treatment) changing the top guesses until the equivalents of Eqs. (26) and (27) were satisfied at the bottom; but calculational instability prevented use of this method. When, instead, the equations were integrated in the direction of the radiant stream, those for B_D and W_D from the top down and those for B_U and W_U from the bottom up, convergence was satisfactory.

This method of predicting furnace performance is limited in scope to that class of furnaces in which gas temperature varies primarily in a single direction, but the class of such furnaces is a large one. The method has apparently had considerable practical use in England.

Note. To allow for the effect of temperature on gas emissivity k was assumed proportional to $1/T_g$ and a assumed constant. From our work with the zone method it is clear that ε_g can be represented over a far wider range of T_g by assuming a to be a function of T and k constant.

Time prevents an exposition of the zone method (or a variation on it, the zone-to-point method), which has been extensively used in MIT studies of furnace performance and is presently being used in a study of interaction between natural convection and radiation in glass-furnace melts.

LITERATURE CITED

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